# 0

# Bridging material

This chapter will remind you how to

- manipulate algebraic expressions and evaluate formulae
- solve linear equations and graph linear equations
- find the midpoint and length of a line segment
- solve simultaneous equations and deal with inequalities
- use circle theorems in problems
- use Pythagoras' theorem and the trigonometric ratios.

## 0.1 Expanding and factorising

To simplify an algebraic expression, you collect **like terms**.

Like terms have the same **base** and the same **index**.

$2x^2$ and $x^2$ are like terms.
They both have base $x$ and index $^2$.

$2x^2$ and $x^2y$ are unlike terms.
They contain different base letters.

**EXAMPLE 1**

Simplify

**a** $3a - 2b + 4a - 3b + c$

**b** $x - x^2 + 2x + 3x^2$

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**a** Collect the like terms: $3a - 2b + 4a - 3b + c$
$= 7a - 5b + c$

$3a$ and $4a$ are like terms.
$-2b$ and $-3b$ are like terms.

**b** Collect the like terms: $x - x^2 + 2x + 3x^2$
$= 3x + 2x^2$

$x$ and $2x$ are like terms.
$-x^2$ and $3x^2$ are like terms.

To **expand** double brackets, you multiply each term in the first bracket by each term in the second bracket.

Expand means multiply out.

$(x + 1)(x + 2) = x^2 + 2x + x + 2$
$= x^2 + 3x + 2$

You may find it helpful to multiply the terms in this order:

| | | |
|---|---|---|
| F | Firsts | $x \times x$ |
| O | Outers | $x \times 2$ |
| I | Inners | $1 \times x$ |
| L | Lasts | $1 \times 2$ |

**EXAMPLE 2**

Expand and simplify

**a** $(x + 4)(x + 5)$ **b** $(2x - 1)(x - 3)$

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**a** $(x + 4)(x + 5)$
$= x^2 + 5x + 4x + 20$
$= x^2 + 9x + 20$

**b** $(2x - 1)(x - 3)$
$= 2x^2 - 6x - x + 3$
$= 2x^2 - 7x + 3$

Remember the rules for multiplying negative terms.
⊕ × ⊕ ≡ ⊕
⊖ × ⊖ ≡ ⊕
⊕ × ⊖ ≡ ⊖
⊖ × ⊕ ≡ ⊖

The reverse of expanding is called **factorising**.

$$\text{e.g.}\quad \underset{\xleftarrow[\text{factorise}]{}}{\overset{\xrightarrow{\text{expand}}}{2(x + 3) = 2x + 6}}$$

To factorise an expression you write it as a product of its factors.

**EXAMPLE 3**

Factorise fully

**a** $2x + 8$

**b** $y^2 - 3y$

**c** $2z^2 + 4z$

---

**a** $2x + 8$
$= 2 \times x + 2 \times 4$
$= 2(x + 4)$

Make sure you take out the **highest common factor** to factorise fully.

**b** $y^2 - 3y$
$= y \times y - 3 \times y$
$= y(y - 3)$

**c** $2z^2 + 4z$
$= 2z \times z + 2z \times 2$
$= 2z(z + 2)$

You can sometimes factorise a quadratic expression into double brackets.

$$\text{e.g.}\quad \underset{\xleftarrow[\text{factorise}]{}}{\overset{\xrightarrow{\text{expand}}}{(x + 2)(x - 4) = x^2 - 2x - 8}}$$

The two numbers in the brackets multiply to give the constant and add to give the **coefficient** of $x$.

In the expression $x^2 - 2x - 8$ the constant is $-8$, the $x$-coefficient is $-2$ and the $x^2$-coefficient is 1.

**EXAMPLE 4**

Factorise fully $x^2 + 2x - 8$

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$x^2 + 2x - 8$

Consider the factor pairs of $-8$:

$-1, 8;\quad 1, -8;\quad -2, 4;\quad 2, -4$

Find the pair whose sum is $+2$:
$4 + -2 = 2$

$x^2 + 2x - 8 = (x - 2)(x + 4)$

Check your answer by expanding the brackets.

The coefficient of $x^2$ is not always equal to 1.

**EXAMPLE 5**

Factorise $4x^2 + 8x + 3$

$4x^2 + 8x + 3$

Factor pairs of 4 are 1, 4; −1, −4; 2, 2 and −2, −2

Factor pairs of 3 are 1, 3 and −1, −3

Discard the negative factors since all the coefficients in the equation are positive.

Use trial and error:

$(x+1)(4x+3) = 4x^2 + 7x + 3$ ✗

$(2x+1)(2x+3) = 4x^2 + 8x + 3$ ✓

Hence $4x^2 + 8x + 3 = (2x+1)(2x+3)$

Check your answer by expanding the brackets.

Expressions in the form $x^2 - a^2$ are called the **difference of two squares (DOTS)**.

You can factorise the **difference of two squares** expressions using $x^2 - a^2 = (x+a)(x-a)$

When you expand $(x + a)(x - a)$ the two $x$ terms cancel.

The **difference of two squares** can be shown by considering a square of length $x$ which contains a smaller square of length $a$.

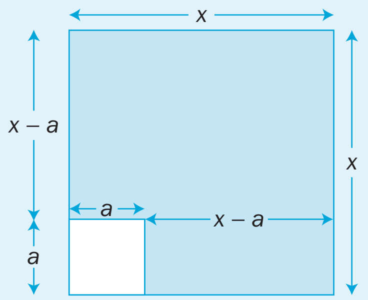
Shaded area $= x^2 - a^2$

But also shaded area

$= x(x-a) + a(x-a)$

$= (x+a)(x-a)$

Hence $x^2 - a^2 = (x+a)(x-a)$



**EXAMPLE 6**

Factorise $x^2 - 25$

$x^2 - 25 = x^2 - 5^2$ so use DOTS:

$x^2 - 25 = (x+5)(x-5)$

Check your answer by expanding the brackets.

You can sometimes simplify algebraic fractions by first factorising.

**EXAMPLE 7**

Simplify the algebraic fraction $\frac{12x - 6x^2}{3x^2}$

Factorise the numerator: $\frac{12x - 6x^2}{3x^2} = \frac{6x(2-x)}{3x^2}$

There is a common factor of 3x on the top and the bottom. $= \frac{2 \times \cancel{3x}(2-x)}{\cancel{3x} \times x}$

$6x = (2)(3x)$

$3x^2 = (x)(3x)$

Cancel down: $= \frac{2(2-x)}{x}$

The simplified fraction is $\frac{2(2-x)}{x}$

You can only cancel by a term which is a factor of both the numerator and the denominator.

**EXAMPLE 8**

Simplify the fraction $\frac{x^2 + 6x + 8}{x + 4}$

Factorise the numerator:

$x^2 + 6x + 8 = (x+2)(x+4)$

Check by multiplying out.

The fraction becomes

$\frac{x^2 + 6x + 8}{x + 4} = \frac{(x+2)\cancel{(x+4)}}{\cancel{(x+4)}}$

Cancel out the common factor $(x + 4)$:

$\frac{x^2 + 6x + 8}{x + 4} = x + 2$

In the next example you have to factorise both the numerator and the denominator to simplify the fraction.

**EXAMPLE 9**

Simplify the fraction $\frac{2x^2 + 5x - 3}{2x^2 + 7x + 3}$

Factorise both the numerator and denominator:

$2x^2 + 5x - 3 = (2x-1)(x+3)$ and $2x^2 + 7x + 3 = (2x+1)(x+3)$

The fraction becomes $\frac{2x^2 + 5x - 3}{2x^2 + 7x + 3} = \frac{(2x-1)\cancel{(x+3)}}{(2x+1)\cancel{(x+3)}}$

Cancel down by the common factor $(x + 3)$:

The simplified fraction is $\frac{2x-1}{2x+1}$

The ‘2x’ terms cannot be cancelled as they are not a common factor.

## Exercise 0.1

**1** Simplify these expressions.

**a** $4a + 3b - 7a - 2$

**b** $m^2 + 2m - n^2 - 2m^2$

**c** $2ab + 5ab - 7ab + ab$

**d** $-pq + 2p^2q^2 + p^2q^2 - pq$

**e** $3 - 2a + 6 - b$

**f** $x^3 - 4 + 4x^3 - 3$

**g** $3(a - b) - 4(a + b)$

**h** $2(1 - 3x) - 2(3 - x)$

**i** $a(a - b) - b(a - b)$

**j** $b(a - b) - a(b - a)$

**2** Expand and simplify these expressions.

**a** $(x + 3)(x + 5)$

**b** $x(x - 4)$

**c** $(x - 3)(x + 5)$

**d** $(2x - 1)(x - 3)$

**e** $(x^2 - 2)(x + 4)$

**f** $(2 - x)(5 - 2x)$

**3** Factorise these expressions.

**a** $4 - 2x$

**b** $3x^2 + 6x$

**c** $x^2 + 6x + 9$

**d** $x^2 - 7x + 12$

**e** $x^2 + 8x - 9$

**f** $x^2 - 4x$

**g** $x^2 - 16$

**h** $2x^2 - 18$

**i** $x^2 - 5x - 36$

**j** $x^2 + 22x - 48$

**4** Factorise these expressions.

**a** $2x^2 + 3x + 1$

**b** $3x^2 + 5x + 2$

**c** $3x^2 - 5x - 2$

**d** $6x^2 + 5x + 1$

**e** $6x^2 + 17x - 3$

**f** $12x^2 - 11x + 2$

**5** Simplify these fractions.

**a** $\dfrac{2x^2 + x}{x}$

**b** $\dfrac{x^2 - x^3}{x^2}$

**c** $\dfrac{3x + 6x^2}{3x}$

**d** $\dfrac{6x + 8x^2}{2x^2}$

**e** $\dfrac{5x - 15x^2}{10}$

**f** $\dfrac{2x^3 + 4x^2 + 6x}{2x}$

**g** $\dfrac{4x^3 + 6x^2}{8x}$

**h** $\dfrac{x^2 - x}{x - 1}$

**6** Simplify these fractions by factorising where possible.

**a** $\dfrac{x^2 + 3x + 2}{x + 2}$

**b** $\dfrac{x^2 - 2x + 1}{x - 1}$

**c** $\dfrac{x^2 - x - 6}{x + 2}$

**d** $\dfrac{x^2 - 2x - 8}{x - 4}$

**e** $\dfrac{x^3 + 3x^2 + 2x}{x^2 + 3x + 2}$

**f** $\dfrac{4x^2 - 4x + 1}{2x - 1}$

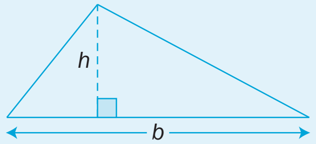
**g** $\dfrac{6x^2 - x - 1}{2x^2 + x - 1}$

**h** $\dfrac{x^2 + 7x + 10}{2x^2 + 11x + 5}$

**i** $\dfrac{2x^2 - 9x + 9}{2x^2 - 11x + 12}$

## 0.2 Formulae

A **formula** is used to express the relationship between two or more variables.



The formula $A = \frac{1}{2}bh$ expresses the relationship between $A$, the area, $b$, the base and $h$, the height of a triangle.

By **substituting** values into a formula you can work out the value of an unknown.

Make sure you use the correct order of operations:
**B**rackets
**I**ndices
**D**ivision
**M**ultiplication
**A**ddition
**S**ubtraction.

**EXAMPLE 1**

Given the values $p = 3$, $q = -5$, $r = 2$ find the value of $T$ in each case.

**a** $T = \frac{p^2 - q}{r}$ **b** $T = \frac{3p}{r} - \frac{2r}{q}$

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**a** $T = \frac{p^2 - q}{r}$

Substitute in the values: $T = \frac{(3)^2 - (-5)}{2}$

Simplify: $= \frac{9 + 5}{2}$

$= \frac{14}{2} = 7$

Take care when substituting negative values.
$-q = -(-5) = 5$

**b** $T = \frac{3p}{r} - \frac{2r}{q}$

Substitute in the values: $T = \frac{3 \times 3}{2} - \frac{2 \times 2}{-5}$

$-\frac{2 \times 2}{-5} = \frac{-4}{-5} = \frac{4}{5}$

Simplify by expressing the fractions with a common denominator:

$$T = \frac{9}{2} + \frac{4}{5}$$

$$= \frac{45 + 8}{10}$$

$$= \frac{53}{10}$$

$$= 5\frac{3}{10}$$

$\frac{9}{2} + \frac{4}{5}$

$= \frac{45}{10} + \frac{8}{10}$

$= \frac{45 + 8}{10}$

Sometimes you will need to **change the subject** of a formula by rearranging its terms.

**EXAMPLE 2**

Make $x$ the subject of the formula $xyz + t = M$

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$xyz + t = M$

Isolate the $x$-term: $xyz = M - t$

Divide both sides of the formula by $yz$: $\frac{xyz}{yz} = \frac{M - t}{yz}$

Simplify: $x = \frac{M - t}{yz}$

Making $x$ the subject means rearranging the formula to express $x$ in terms of the other variables.

**EXAMPLE 3**

Make $y$ the subject of the formula $\frac{h}{x} - \frac{p}{y} = 1$

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$\frac{h}{x} - \frac{p}{y} = 1$

Isolate the $y$-term: $-\frac{p}{y} = 1 - \frac{h}{x}$

Multiply both sides by −1 and rearrange: $\frac{p}{y} = \frac{h}{x} - 1$

$\frac{p}{y} = \frac{h - x}{x}$

Remember to multiply **all** terms by −1.

Invert both fractions: $\frac{y}{p} = \frac{x}{h - x}$

Turn the fractions upside down so that $y$ is on top of its fraction.

Multiply both sides by $p$: $p \times \frac{y}{p} = \frac{px}{h - x}$

You want to get $y$ on its own.

Simplify: $y = \frac{px}{h - x}$

### Exercise 0.2

**1** Given the values $a = 4$, $b = 9$ and $c = 3$, find the value of $P$ in each case.

**a** $P = 3a^2 - bc$

**b** $P = a - b \div c$

**c** $P = \sqrt{(a^2 + c^2)}$

**d** $P = \sqrt{a}\sqrt{b}$

**e** $P = (a + b)(b - c)$

**f** $P = \frac{3a + 2b}{c}$

**g** $P = \sqrt{\frac{a}{b}} + \frac{a}{c}$

**h** $P = \sqrt{\frac{bc^2}{a}}$

**i** $P = \frac{b}{a} - \frac{a}{c}$

**j** $P = \left(\frac{b + c}{a}\right)^2$

**2** Use substitution to find the value of the subject of each formula.

**a** $A = \pi r^2; \quad \pi = 3, \ r = 0.4$

**b** $A = 4\pi r^2; \quad r = \frac{1}{2}, \ \pi = 3$

**c** $L = 2(a + b); \quad a = 0.07, \ b = 0.7$

**d** $I = \frac{PTR}{100}; \quad P = 160, \ T = 3, \ R = 5$

**e** $V = \pi r^3; \quad r = 4, \ \pi = 3$

**f** $l = \sqrt{(a^2 - b^2)}; \quad a = 13, \ b = 12$

**g** $A = \pi rl; \quad r = 0.2, \ l = 1.4, \ \pi = 3$

**h** $V = \frac{4}{3}\pi r^3; \quad \pi = 3, \ r = 3$

**i** $T = 2\pi\sqrt{\frac{l}{g}}; \quad \pi = 3, \ g = 32, \ l = 2$

**j** $f = \frac{uv}{u + v}; \quad u = \frac{1}{2}, \ v = \frac{1}{3}$

**3** Make $x$ the subject of each formula.

**a** $y = 3x - 4$

**b** $2y + x - 3 = 0$

**c** $y = mx + c$

**d** $3(x + y) = 5$

**e** $ax + by + c = 0$

**f** $(x - 1)(y + 2) = 3$

**g** $\frac{1}{x} + \frac{1}{y} = 1$

**h** $3x^2 + y - 4 = 0$

**i** $\frac{2}{x + 1} = y$

**j** $\frac{1}{x + 1} - \frac{1}{y - 1} = 1$

**4** Find the value of the unknown in each formula.

**a** $4a - 3b + 2c = 0; \quad a = 3, \ b = 5; \quad$ find $c$

**b** $x^2 + y^2 = r^2; \quad x = 16, \ r = 20; \quad$ find $y$

**c** $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}; \quad r = 5, \ p = 6; \quad$ find $q$

**d** $\frac{3}{m} - \frac{n}{4} = \frac{1}{p}; \quad m = 2, \ n = 3; \quad$ find $p$

**e** $(2 + p)(3 + q) = r; \quad p = \frac{1}{4}, \ r = 2; \quad$ find $q$

**f** $\frac{2a}{b} - \frac{c}{3b} = c; \quad a = 2, \ c = 4; \quad$ find $b$

**g** $\frac{4}{3}m(n - 1) = p; \quad m = 0.5, \ p = 1.5; \quad$ find $n$

**h** $\frac{4 - r}{3 - s} = \frac{r}{s}; \quad s = -3; \quad$ find $r$

**5** **a** In vertical motion under gravity the distance, $s$, which a stone falls is related to its initial speed, $u$, time, $t$, and acceleration, $a$, due to gravity where

$$s = ut + \frac{1}{2}at^2$$

Make $u$ the subject of the formula.
Find the value of $u$ when $s = 200$, $a = 10$ and $t = 5$.

**b** A force acts on a particle of mass, $m$, and the velocity of the particle changes from $u$ to a new velocity, $v$.
The impulse of the force is measured by the formula

$$I = mv - mu$$

Make $m$ the subject of the formula.
Find the value of $m$ when $I = 240$, $u = 0.6$ and $v = 1.8$.

## 0.3 Solving linear equations

**Linear equations** involve a variable, such as $x$.
You can solve a linear equation to find a unique solution.

To solve a linear equation, you find the value of its variable.

EXAMPLE 1

Find the value of $x$ where $3x - 1 = 8$

Isolate the $x$-term: $3x = 8 + 1$

Simplify: $3x = 9$

Divide both sides by 3: $\frac{3x}{3} = \frac{9}{3}$

Hence $x = 3$

You want to get $x$ on its own.

Check your answer by substituting for $x$ in the original equation.

EXAMPLE 2

Solve the equation $5x - 2 = 3x + 4$

Collect the $x$-terms on one side and the constants on the other side:

$$5x - 3x = 4 + 2$$

Collect like terms: $2x = 6$

Divide both sides by 2: $\frac{2x}{2} = \frac{6}{2}$

Hence $x = 3$

Check your answer by substituting for $x$ in the original equation.

Sometimes you will need to simplify an equation before solving it.

EXAMPLE 3

Solve the equation $2(4t - 1) - 3(t - 2) = 0$

Expand the brackets: $8t - 2 - 3t + 6 = 0$

Collect like terms: $5t + 4 = 0$

Rearrange: $5t = -4$

Divide both sides by 5: $\frac{5t}{5} = -\frac{4}{5}$

Hence $t = -\frac{4}{5}$

Take care when multiplying negative terms.

Check your answer by substituting for $t$ in the original equation.

Linear equations may involve algebraic fractions.

EXAMPLE 4

Solve the equation $\frac{x+1}{3} + 1 = \frac{x}{2}$

Combine the terms on the LHS: $\frac{x+1+3}{3} = \frac{x}{2}$

Multiply both sides by 6: $2(x + 4) = 3x$

Expand: $2x + 8 = 3x$

Collect like terms and rearrange: $-x = -8$

Divide both sides by −1: $x = 8$

6 is the LCM of 2 and 3.

Check your answer by substituting $x = 8$ into the original equation.

### Exercise 0.3

**1** Find the value of $x$ for each equation.

**a** $2x + 5 = 9$ **b** $3 - 4x = -9$ **c** $7 = 2 - 2x$

**d** $\frac{1}{2}x + 1 = 3\frac{1}{2}$ **e** $-\frac{2x}{3} - 2 = \frac{4}{3}$ **f** $1 + \frac{x}{3} = -\frac{2}{3}$

**g** $3x - 2 = x + 6$ **h** $2 - x = 4x - 8$ **i** $5x - 4 = 2x - 16$

**j** $\frac{x}{2} + \frac{x}{3} = 1$ **k** $3x + 2 + 7x - 4 = -4x$ **l** $\frac{4x}{5} = -2 + \frac{2x}{3}$

**2** Find the value of the unknown in each equation.

**a** $3(t - 2) = 6$ **b** $5(2t + 1) = -15$ **c** $2(1 - 3p) = 0$

**d** $1 = 3(y - 1)$ **e** $-2(r - 3) = 4$ **f** $3 - 4(x - 1) = 0$

**g** $3(y + 2) = 2(y - 1)$ **h** $2(2t - 1) - 3(t + 1) = 0$ **i** $4 - 2(t + 3) = 0$

**j** $4(1 - r) - 3(r - 1) = 0$ **k** $5(2x - 1) = -2(5x + 1)$ **l** $\frac{1}{2}(x + 3) - (1 - x) = 1$

**3** Solve these equations.

**a** $\frac{x}{3} - \frac{2x}{5} = 2$ **b** $\frac{x+1}{2} = \frac{2x}{5}$

**c** $\frac{3}{x+1} = \frac{2}{x-1}$ **d** $3 + \frac{2-x}{2} = \frac{2x+1}{3}$

**e** $\frac{3x}{2} = \frac{x}{3} + 1$ **f** $\frac{x+2}{3} + \frac{2x-3}{4} = 2$

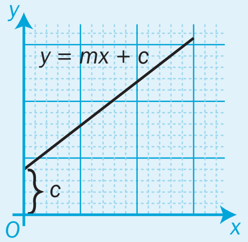
**g** $\frac{x+1}{3x} = \frac{x+2}{3x+1}$ **h** $\frac{x(1+2x)}{x-1} = 2x - 1$

## 0.4 Plotting graphs

### Linear functions

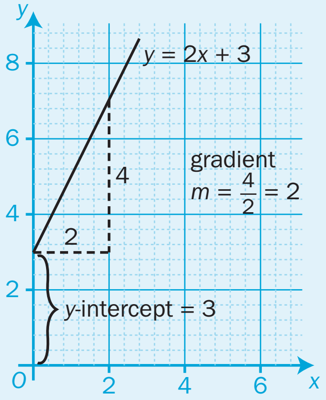
**Linear functions** can be written in the form
$y = mx + c$ where $m$ and $c$ are constants.

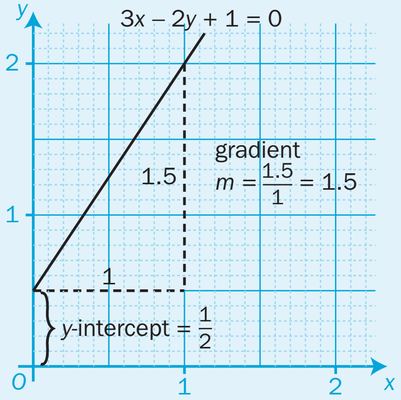
A linear function is represented graphically by a straight line.
$m$ is the gradient and $c$ is the $y$-intercept of the graph.

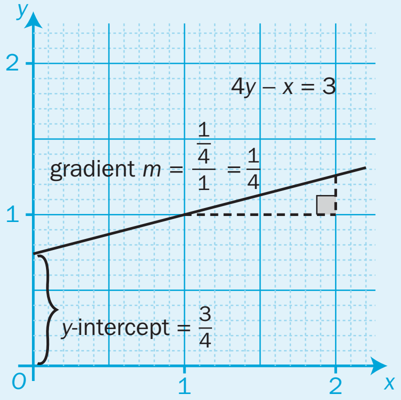


Here are some examples of linear functions.

| | | |
|---|---|---|
| $y = 2x + 3$ | $3x - 2y + 1 = 0$<br>so $y = \frac{3}{2}x + \frac{1}{2}$ | $4y - x = 3$<br>so $y = \frac{x}{4} + \frac{3}{4}$ |
| gradient $= 2$ | gradient $= \frac{3}{2}$ | gradient $= \frac{1}{4}$ $\quad \frac{x}{4} = \frac{1}{4} \times x$ |
| $y$-intercept $= 3$ | $y$-intercept $= \frac{1}{2}$ | $y$-intercept $= \frac{3}{4}$ |







To find the $y$-axis crossing, substitute $x = 0$ into the linear equation and solve for $y$.
To find the $x$-axis crossing, substitute $y = 0$ into the linear equation and solve for $x$.

**EXAMPLE 1**

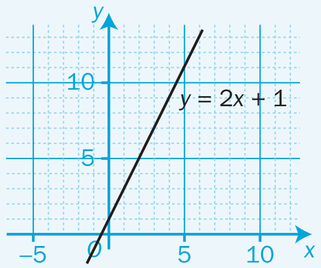
Draw the graph of $y = 2x + 1$

$y = 2x + 1$

Make a table of values:

| | | | |
|---|---|---|---|
| $x$ | 0 | 2 | 4 |
| $y$ | 1 | 5 | 9 |

Use your table of values to draw the straight line graph.



Include at least three values in your table. Use two values to draw the line and the third value to check.

Check with the equation:
gradient = 2; $y$-intercept = 1

**EXAMPLE 2**

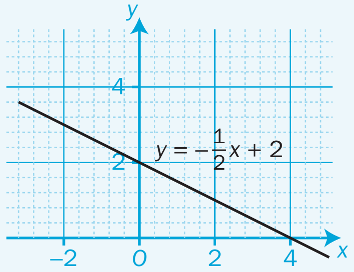
Plot the graph of $y = -\frac{1}{2}x + 2$

$y = -\frac{1}{2}x + 2$

Make a table of values:

| | | | |
|---|---|---|---|
| $x$ | -2 | 0 | 2 |
| $y$ | 3 | 2 | 1 |

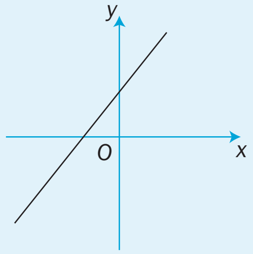
Use your table of values to draw the straight line graph.



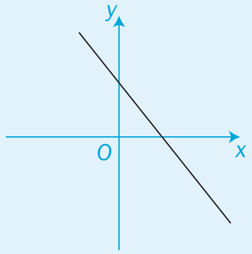
Check with the equation:
gradient $= -\frac{1}{2}$
$y$-intercept = 2

Notice that the line graphs in Examples 1 and 2 slope in different directions.

The direction of slope of a straight line relates to the sign of the gradient.



positive gradient



negative gradient

Sometimes you will need to rearrange the equation into the form $y = mx + c$ before you can plot the line graph.

**EXAMPLE 3**

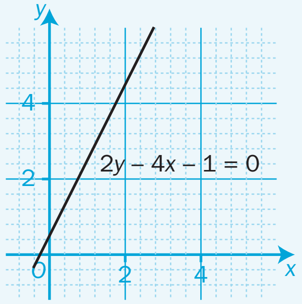
Plot the graph of $2y - 4x - 1 = 0$

$$2y - 4x - 1 = 0$$

Isolate the $y$-term: $2y = 4x + 1$

Divide by 2: $\frac{2y}{2} = \frac{4x}{2} + \frac{1}{2}$

Cancel down: $y = 2x + \frac{1}{2}$



Make a table of values and use it to draw the graph:

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | 0.5 | 4.5 | 8.5 |

Check with the equation:
gradient = 2
$y$-intercept = $\frac{1}{2}$

Alternatively, you could use the 'cover up' method.

Function notation is often used to describe an equation.

**EXAMPLE 4**

Plot the graphs of $y = f(x)$ and $y = -f(x)$ where $f(x) = -2x$

Let $y = f(x)$, then $y = -2x$

Make a table of values for $y = f(x) = -2x$:

| $x$ | −2 | 0 | 2 |
|---|---|---|---|
| $y$ | 4 | 0 | −4 |

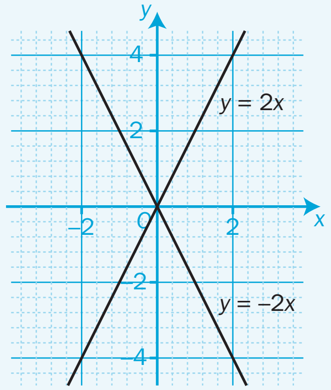
$y = -f(x)$ gives $y = 2x$

Make a table of values for $y = -f(x) = 2x$:

| $x$ | −2 | 0 | 2 |
|---|---|---|---|
| $y$ | −4 | 0 | 4 |

Take care with negative signs.
$-f(x) = -(-2x) = 2x$

Use your tables of values to draw the straight line graphs:



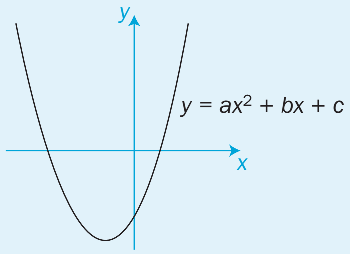
Gradient of f($x$) is −2.
Gradient of −f($x$) is 2.

## Quadratic functions

The general form of a **quadratic function** is

$y = ax^2 + bx + c$

where $a$, $b$ and $c$ are constant values and $a \neq 0$.



The graph of a quadratic function is in the shape of a

You can draw the graph of a quadratic function by constructing a table of values.
Use suitable values of $x$ to include the important features.

**EXAMPLE 1**

Plot the graph of the function $y = x^2 + 1$ taking values of $x$ from −3 to 3.

Construct a table of values:

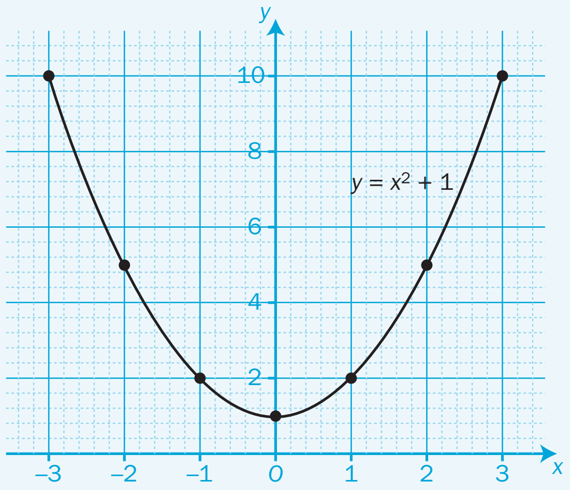
| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $x^2$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |
| +1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $y$ | 10 | 5 | 2 | 1 | 2 | 5 | 10 |

You may find it easier to calculate each term of the equation separately first.

Use your table of values to draw the graph:

The curve intersects the $y$-axis when $x = 0$.
When $x = 0$, $y = 1$ so the coordinates of this point are (0, 1).

The curve does not intersect the $x$-axis since $y \neq 0$ for any value of $x$.



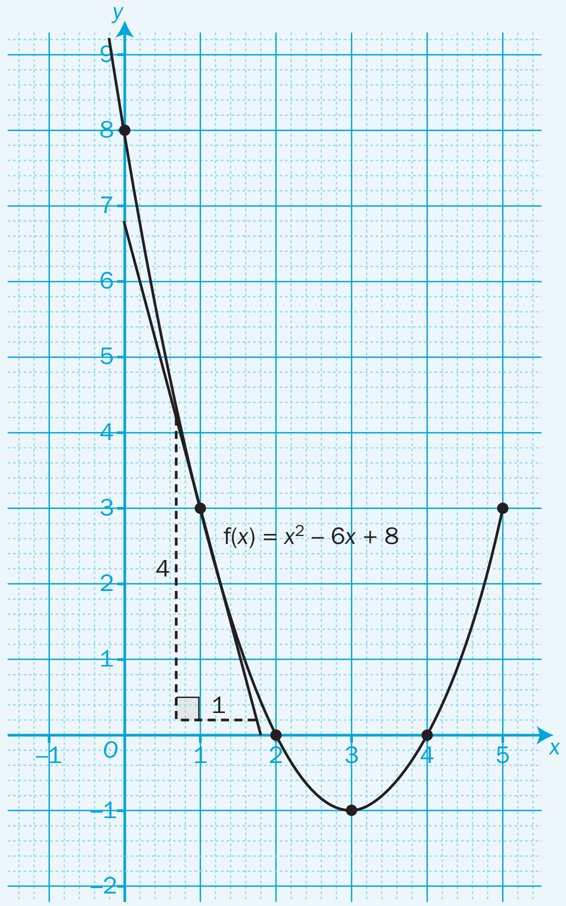
**EXAMPLE 2**

Plot the graph of the function $f(x) = x^2 - 6x + 8$ taking values of $x$ from 0 to 5.
Find the coordinates of the points where the curve intersects the $x$-axis and the $y$-axis.
Estimate the gradient of the tangent at the point where $x = 1$.

Construct a table of values:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x^2$ | 0 | 1 | 4 | 9 | 16 | 25 |
| $-6x$ | 0 | −6 | −12 | −18 | −24 | −30 |
| $+8$ | 8 | 8 | 8 | 8 | 8 | 8 |
| $y$ | 8 | 3 | 0 | −1 | 0 | 3 |

Use your table of values to plot the graph:



$f(0) = 8$ so the curve intersects the $y$-axis at (0, 8).

$f(2) = 0$ and $f(4) = 0$ so the curve intersects the $x$-axis at (2, 0) and (4, 0).

To measure a gradient on a curve: draw a tangent, then measure the gradient of the tangent.

From the graph, the gradient at $x = 1$ is roughly −4.

## Exercise 0.4

**1** Plot the graph of each function taking the given values of $x$.

- **a** $y = x - 3$, $(x = -2 \text{ to } 4)$
- **b** $y = -x + 4$, $(x = -2 \text{ to } 5)$
- **c** $y = 2x - 3$, $(x = -1 \text{ to } 5)$
- **d** $y = -3x + 5$, $(x = -2 \text{ to } 3)$

**2** Plot the graph of each function taking $x$-values from −2 to 4.

- **a** $f(x) = \frac{1}{2}x - 2$
- **b** $f(x) = -2x + 4$
- **c** $f(x) = -x + 2.5$
- **d** $f(x) = 5x - 10$

**3** Make $y$ the subject of each equation and plot the graphs taking suitable values of $x$.

- **a** $2y - 3x + 2 = 0$
- **b** $x + 2y - 8 = 0$
- **c** $5 = 4x - 2y$

**4** Plot the graph of each equation stating the value of the gradient.

- **a** $\frac{1}{2}(y + 1) = \frac{1}{2}(x - 1)$
- **b** $y = 2f(x)$, where $f(x) = 1 - x$
- **c** $y = -2$

**5** **a** Copy and complete the table of values for $y = x^2 - 4x$ for $x = -1$ to $x = 5$.

| $x$ | −1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| $x^2$ | | | | | | | |
| $-4x$ | | | | | | | |
| $y$ | | | | | | | |

**b** Write down the coordinates of the points where the curve intersects the
- **i** $x$-axis
- **ii** $y$-axis.

**6** **a** Copy and complete the table of values for $y = x^2 - 2x - 8$ for $x = -3$ to $x = 5$.

| $x$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|
| $x^2$ | | | | | | | | | |
| $-2x$ | | | | | | | | | |
| $-8$ | | | | | | | | | |
| $y$ | | | | | | | | | |

**b** Plot the graph of the quadratic.

**7** **a** Copy and complete the table of values for $y = 2 + x - x^2$ for $x = -2$ to $x = 3$.

| $x$ | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $-x^2$ | | | | | | |
| $x$ | | | | | | |
| $+2$ | | | | | | |
| $y$ | | | | | | |

**b** Plot the curve.

**c** Find the coordinates of the points where the curve intersects the
**i** $x$-axis
**ii** $y$-axis.

**8** **a** Plot the graphs of $y = x^2 + 2$ and $y = x^2 - 3$ on the same axes taking values of $x$ from –3 to 3.

**b** What do you notice about the relative positions of these curves?

**9** **a** Plot the graphs of $y = -x^2 + 3$ and $y = -x^2 - 4$ on the same axes taking values of $x$ from –3 to 3.

**b** What do you notice about the relative positions of these curves?

**c** Compare these curves to the ones drawn in question 8.

**10** **a** Plot the graphs of $y = x^2$, $y = 2x^2$ and $y = \frac{1}{2}x^2$ on the same axes for $x = -3$ to $x = 3$.

**b** What similarities or differences do you notice between the curves?

**11** **a** Plot the curve of $y = 3x^2 + 7x - 20$ taking values of $x$ from –5 to 3.

**b** Write down the coordinates of the points where the curve intersects the
**i** $x$-axis
**ii** $y$-axis.

**c** Use the graph to estimate the coordinates of its **turning point**.

This is the point at which the graph changes direction.

**d** Estimate the gradient of the tangent at the point where $x = -2$.

**12** **a** Plot the graph of the function

$$f(x) = x^2 - x - 6$$

taking values of $x$ from –3 to 4.

**b** Find the values of $x$ for which $f(x) = 0$

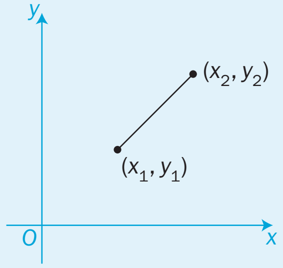
**c** Plot the graph of $g(x) = x - 5$ on the same axes.

**d** Estimate the values for which $f(x) = g(x)$ giving your values of $x$ to 1 d.p.
How many values are there?

## 0.5 The midpoint and length of a line segment

The midpoint of the straight line joining two points $(x_1, y_1)$ and $(x_2, y_2)$ has coordinates

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$



**EXAMPLE 1**

Points $A$ and $B$ have coordinates $(1, 4)$ and $(-2, -3)$ respectively.
Find the midpoint of the straight line $AB$.

Use the formula to find the coordinates of the midpoint:

The midpoint is at $\left(\frac{1+(-2)}{2}, \frac{4+(-3)}{2}\right)$

$$= \left(-\frac{1}{2}, \frac{1}{2}\right)$$

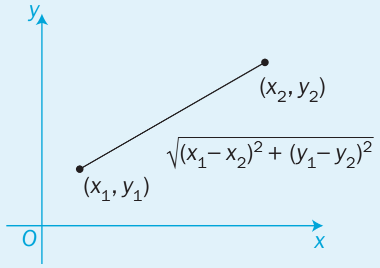
Take care with the negative signs.

You can sketch a diagram to check if your answer is sensible.

You can use Pythagoras' Theorem to find the length of a line joining two points.

The length of a straight line segment joining two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$



**EXAMPLE 2**

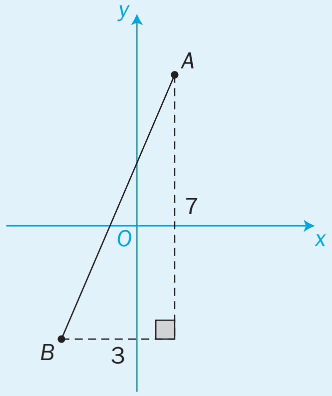
Find the length of the line $AB$ in Example 1.

Sketch a diagram, labelling the unknown side.

Use Pythagoras' Theorem to find the missing length, $AB$:

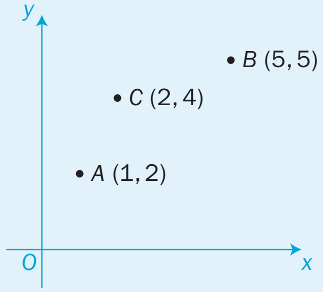
$$AB^2 = 3^2 + 7^2$$
$$= 9 + 49$$
$$= 58$$

Hence length of line $AB = \sqrt{58}$



### Exercise 0.5

1 The diagram shows three points $A$, $B$ and $C$.



Find

- **a** the midpoint of the line joining the points $A$ and $B$
- **b** the length of the line joining the points $A$ and $C$
- **c** the midpoint of the line joining points $B$ and $C$.

Show your working in each case.

2 Find **i** the midpoint
**ii** the length
of the line joining each pair of points.

- **a** $(0, 0)$ and $(2, 3)$
- **b** $(1, 1)$ and $(4, 2)$
- **c** $(1, -1)$ and $(6, 5)$
- **d** $(0, 2)$ and $(4, 1)$
- **e** $(-2, 4)$ and $(1, 1)$
- **f** $(-2, 0)$ and $(1, 3)$

3 A line which joins two points $A$ and $B$ has a midpoint with coordinates $\left(\frac{7}{2}, \frac{3}{2}\right)$.
Use a suitable formula to find the coordinates of point $B$ if

- **a** point $A$ is at $(1, 1)$
- **b** point $A$ is at $(3, -1)$.

4 Two points $A$ and $B$ are joined by a line.
Point $A$ has coordinates $(-1, -1)$.
The midpoint of the line $AB$ has coordinates $(1, 2)$.
Show that the length of line $AB$ is $2\sqrt{13}$.

## 0.6 Simultaneous equations

You can solve a pair of **simultaneous equations** by eliminating one of the variables.

The solution gives the values of the two variables which satisfy both equations at the same time.

EXAMPLE 1

Solve the simultaneous equations for $x$ and $y$.

$2x - 3y = 13, \quad 5x + 3y = 1$

Aim to eliminate either $x$ or $y$.

Label the equations and add them:

$$\begin{array}{ccccl} 2x & - & 3y & = 13 & (1) \\ + & & + & + & \\ 5x & + & 3y & = 1 & (2) \end{array}$$

Labelling the equations will help you to organise your working.

The coefficient of $y$ in (1) is –3.
The coefficient of $y$ in (2) is 3.

$-3y + 3y = 0$, so $y$ can be eliminated by adding the equations.

$$2x + 5x - 3y + 3y = 13 + 1$$

Collect like terms: $7x + 0 = 14$

Divide by 7: $x = 2$

Substitute $x = 2$ into equation (1) to find the value of $y$:

$$2(2) - 3y = 13$$
$$4 - 3y = 13$$

Either equation can be used. Choose the one which involves the easiest substitution.

Rearrange and simplify:

$$-3y = 13 - 4$$
$$-3y = 9$$

Divide by –3: $y = -3$

$\frac{-3y}{-3} = \frac{9}{-3}$

Hence the solution is $x = 2$ and $y = -3$.

Check by substituting for $x$ and $y$ in the original equations.

Sometimes you will need to multiply one of the equations by a constant to make the coefficients of either $x$ or $y$ the same.

EXAMPLE 2

Find $x$ and $y$ when $3x + y = -5$ and $4x - 3y = -11$

Rewrite and label the equations:

$$3x + y = -5 \quad (1)$$
$$4x - 3y = -11 \quad (2)$$

It helps to write one equation under the other so that you can compare the $x$- and $y$-coefficients.

Multiply equation (1) by 3 to make the coefficients of $y$ the same, but with opposite signs. Label this equation (3).

The coefficient of $y$ in (1) is 1.
The coefficient of $y$ in (2) is –3.

3 × (1): $9x + 3y = -15 \quad (3)$

Remember to multiply all of the terms by 3.

Compare with equation (2): $4x - 3y = -11 \quad (2)$

Add equations (2) and (3): $9x + 4x = -15 - 11$

Simplify: $13x = -26$

Divide by 13: $x = -2$

Substitute $x = -2$ into equation (1):

$$3(-2) + y = -5$$
$$-6 + y = -5$$

Rearrange:

$$y = -5 + 6$$
$$y = 1$$

Hence the solution is $x = -2$ and $y = 1$.

Check by substituting the values of $x$ and $y$ into the original equations.

Sometimes both equations need to be multiplied by a constant so that one variable can be eliminated.

**EXAMPLE 3**

Solve the simultaneous equations

$3y + 4x = -4$ and $2y + 6x = -1$

Rewrite and label the equations: $3y + 4x = -4 \quad (1)$

$2y + 6x = -1 \quad (2)$

Multiply equation (1) by 2 and equation (2) by 3:

$$3y + 4x = -4 \rightarrow 2 \times (1) \rightarrow 6y + \phantom{1}8x = -8 \quad (3)$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad - \quad\; - \quad\; -$$

$$2y + 6x = -1 \rightarrow 3 \times (2) \rightarrow 6y + 18x = -3 \quad (4)$$

Coefficient of $y$ in (1) is 3.
Coefficient of $y$ in (2) is 2.
LCM(3, 2) = **6**.
Make the coefficient of $y$ in each equation **6**.

Both equations have $y$ coefficient 6 so eliminate by subtracting:
$6y - 6y = 0$

Equation (3) − equation (4): $0y + 8x - 18x = -8 - (-3)$

$$-10x = -8 + 3$$

$$-10x = -5$$

Subtract every term in equation (4) from equation (3).
Take care with the signs.

Divide by −10: $x = \frac{-5}{-10}$

$$x = \frac{1}{2}$$

Substitute $x = \frac{1}{2}$ into equation (1): $3y + 4x = -4$

$$3y + 4\left(\frac{1}{2}\right) = -4$$

Simplify: $3y + 2 = -4$

$$3y = -6$$

Divide by 3: $y = -2$

Hence the solution is $x = \frac{1}{2}$ and $y = -2$.

Check by substituting the values of $x$ and $y$ into the original equations.

You can solve a pair of simultaneous equations by **substitution**.

**EXAMPLE 4**

Solve by substitution $y = 3x + 4$, $4x + y = -3$

Label the equations: $y = 3x + 4 \quad (1)$

$4x + y = -3 \quad (2)$

Substitute $y = 3x + 4$ in equation (2):

$$4x + y = -3$$

$$4x + (3x + 4) = -3$$

Simplify: $7x + 4 = -3$

Rearrange: $7x = -3 - 4$

$$x = -1$$

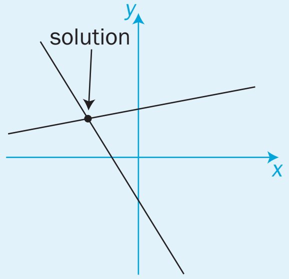
Substitute $x = -1$ into equation (1) to find the value of $y$:

$$y = 3x + 4$$

$$y = -3 + 4 = 1$$

The solution is $x = -1$ and $y = 1$.

You can also **estimate** the solution to a pair of simultaneous equations by plotting their graphs.



**EXAMPLE 5**

Solve graphically $2x + y = 7$, $y = x + 1$

Make a table of values for each equation:

$y = x + 1$

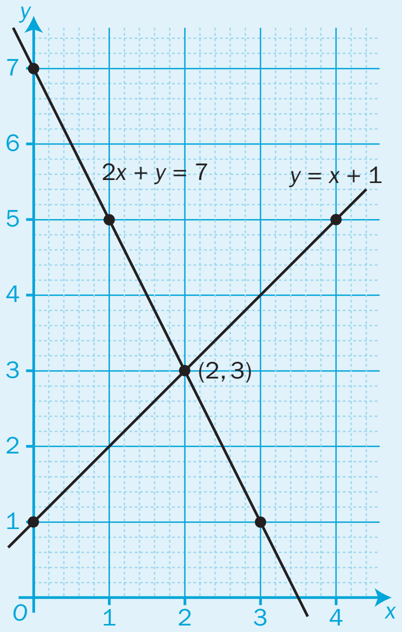
| | | | |
|---|---|---|---|
| $x$ | 0 | 2 | 4 |
| $y$ | 1 | 3 | 5 |

$2x + y = 7$

| | | | |
|---|---|---|---|
| $x$ | 0 | 1 | 3 |
| $y$ | 7 | 5 | 1 |

Use your tables to draw the graphs on the same pair of axes.

The point of intersection has coordinates (2, 3) – this gives the solution $x = 2$ and $y = 3$.



## Exercise 0.6

**1** In each pair of simultaneous equations eliminate one of the unknowns by adding or subtracting the equations.
Hence find the values of $x$ and $y$.

**a** $x + 2y = 3$
$x + y = 2$

**b** $2x - y = 5$
$x + y = 1$

**c** $x + 2y = 5$
$3x + 2y = 3$

**d** $2x - 4y = 6$
$2x + y = 8\frac{1}{2}$

**e** $2y - 3x = 12$
$x - 2y = -8$

**f** $4x - y = 3$
$2y - 4x = -4$

**g** $2y - x = 8$
$3x - 2y = -14$

**h** $3x - y = 7$
$y - 2x = -3$

**i** $4x - 3y = 1$
$4x - 5y = 7$

**j** $2x + y = 5$
$4x + y = 12$

**2** In each pair of simultaneous equations multiply one equation by a suitable constant then add or subtract the equations to eliminate an unknown.
Hence find the values of $x$ and $y$.

**a** $x + 3y = 7$
$2x - y = 0$

**b** $2x - y = 5$
$3x - 2y = 8$

**c** $3y + 2x = -1$
$y - 3x = -15$

**d** $3x + 2y = -10$
$2x - 4y = 4$

**e** $4x - y = 4$
$2x - 2y = 5$

**f** $3y + 4x = 15$
$2y - x = 10$

**g** $2x + y = 6$
$5x + 3y = 16\frac{1}{2}$

**h** $4x - 3y = -7$
$12x - y = -5$

**i** $2y - 3x = 8$
$2x - y = -5\frac{1}{2}$

**j** $6x + 3y = 3$
$4y + 2x = -20$

**3** Use the method of substitution to solve these pairs of equations.

**a** $x + 2y = 8$
$2x + 3y = 14$

**b** $x + 2y = 1$
$2x + 3y = 4$

**c** $4x + y = 14$
$x + 5y = 13$

**d** $c + d = 4$
$2c + d = 5$

**e** $3p - q = 5$
$2p + 5q = 7$

**f** $2y = 4 + x$
$6x - 5y = 18$

**4** Plot the graphs of each of these simultaneous equations on the same diagram and solve them for $x$ and $y$.

**a** $y = 2x + 1$
$y = -2x + 1$

**b** $y = x + 3$
$y = -2x - 3$

**c** $y = \frac{1}{2}x + 1$
$y = 2x - 2$

**d** $y = x - 4$
$y = 2x - 3$

**5** Estimate solutions to these simultaneous equations using a graphical method.

**a** $2y - x = -1$
$y - x = 2$

**b** $3y + 4 = 3x$
$y + x = 2$

**c** $2y = 3x - 2$
$y + 2x = 4$

**d** $2y + 1 = 5x$
$3x + 2y = 7$

**6** Solve these problems by forming and solving simultaneous equations.

**a** When a number $x$ is subtracted from twice the number $y$ the answer is 2.
Three times $y$ subtracted from four times $x$ is 17.
Set up two equations and solve simultaneously to find $x$ and $y$.

**b** Two packs of sandwiches and a cup of coffee cost £4.20 at Jean's Café whilst three packs of sandwiches and two cups of coffee cost £6.70.
Find, by setting up two simultaneous equations, the cost of a pack of sandwiches and the cost of a cup of coffee.

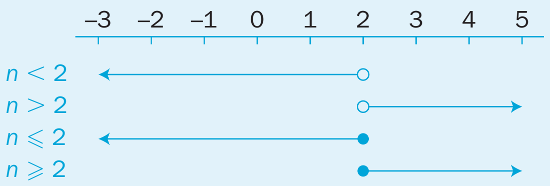
**c** At a theme park the entrance fees are £$x$ for adults and £$y$ for children.
A family of two adults and two children pay £74 and a party of three adults and five children pay £141. Find the admission charge per person.

## 0.7 Inequalities

The four symbols used to describe an inequality are
$<$ $>$ $\leqslant$ and $\geqslant$

$n < 2$ means the number $n$ is less than 2
$n > 2$ means the number $n$ is greater than 2
$n \leqslant 2$ means the number $n$ is less than or equal to 2
$n \geqslant 2$ means the number $n$ is greater than or equal to 2

Inequalities can be shown on a number line.

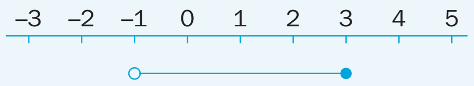


○ means exclude the value
● means include the value

You can use a combination of inequality signs to define a set of values.

**EXAMPLE 1**

On a number line show the set of values satisfied by the inequality $-1 < n \leqslant 3$.

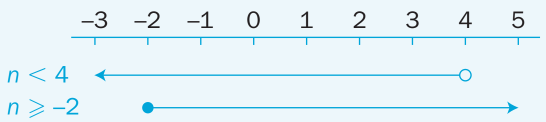


The inequality indicates that $n$ is greater than –1 and less than or equal to 3.

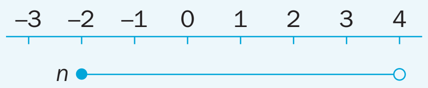
You can combine two different inequalities to define a set of solutions.

**EXAMPLE 2**

On a number line show the solution set satisfied by both $n < 4$ and $n \geqslant -2$.



so the set of solutions for $n$ is $-2 \leqslant n < 4$



$n$ is strictly less than 4 and greater than or equal to –2.
Show each inequality on a number line and then combine to give the final set of solutions.

Sometimes you will need to simplify an inequality before you can show the set of solutions on a number line.

**EXAMPLE 3**

Simplify $-4 < 3x + 2 \leqslant 11$ and show the solution set on a number line.

$-4 < 3x + 2 \leqslant 11$

The inequality has two parts. Solve them separately.

Solve the LH part: $-4 < 3x + 2$

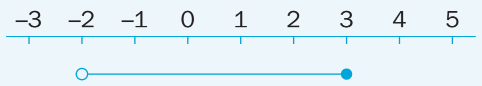
Subtract 2 from each part: $-4 - 2 < 3x + 2 - 2$

Simplify: $-6 < 3x$
Divide by 3: $-2 < x$
Solve the RH part: $3x + 2 \leqslant 11$
$3x + 2 - 2 \leqslant 11 - 2$
$3x \leqslant 9$
$x \leqslant 3$
$\therefore\ -2 < x \leqslant 3$

You need to get $x$ on its own.

$\therefore$ means 'therefore'

Represent the solution $-2 < x \leqslant 3$ on a number line:



$-2 < x \leqslant 3$

Remember
○ means exclude the value
● means include the value

**EXAMPLE 4**

Simplify $2 \leqslant 5 - x < 4$ and show the solution range on a number line.

$2 \leqslant 5 - x < 4$

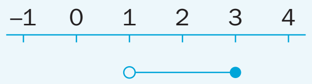
LH part: $2 \leqslant 5 - x$
$2 - 5 \leqslant 5 - x - 5$

$-3 \leqslant -x$
Divide by –1: $3 \geqslant x$
RH part: $5 - x < 4$
$5 - x - 5 < 4 - 5$
$-x < -1$
$x > 1$
$\therefore\ 1 < x \leqslant 3$

Solve each part separately.

Remember that dividing or multiplying by a negative number reverses the sign of the inequality.

Represent the solution $1 < x \leqslant 3$ on a number line:

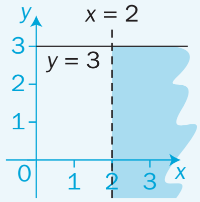


You can show the solution set of an inequality involving two variables as a region on a graph.

- Use shading to show where the solutions lie.
- Use continuous lines to represent the boundary of $\leqslant$ or $\geqslant$ and dashed lines to represent the boundary of $<$ or $>$.

**EXAMPLE 5**

Shade the region on a graph to show the values which satisfy $x > 2$ and $y \leqslant 3$.



Draw the graphs of $x = 2$ and $y = 3$ on the same axes.

$x > 2$, so use a dashed line for $x = 2$.

$y \leqslant 3$ so use a continuous line for $y = 3$.

$x > 2$ so the required values for $x$ are to the right of $x = 2$.

$y \leqslant 3$ so the required values for $y$ are on or below $y = 3$.

Shade the required region.

Read the question carefully. Sometimes you may be asked to shade the unrequired region.

Sometimes you will need to simplify the inequalities before you can show the solution set on a graph.

**EXAMPLE 6**

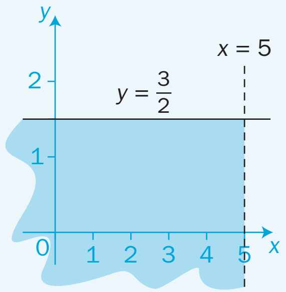
Shade the region on a graph to show the values which satisfy $x - 3 < 2$ and $2y \leqslant 3$.

Simplify: $x - 3 < 2$

$x < 5$

$2y \leqslant 3$

$y \leqslant \frac{3}{2}$



$x < 5$ so use a dashed line for $x = 5$.

$y \leqslant \frac{3}{2}$ so use a continuous line for $y = \frac{3}{2}$.

$x < 5$ so the required values for $x$ are to the left of $x = 5$.

$y \leqslant \frac{3}{2}$ so the required values for $y$ are on or below $y = \frac{3}{2}$.

Shade the required region.

## Exercise 0.7

**1** Use a number line to show the solutions of these inequalities.

**a** $n < 3$
**b** $n \geqslant -1$
**c** $n > 0$ and $n < 4$
**d** $n \geqslant -2, n < 2$
**e** $n < -1, n \geqslant -3$
**f** $-1 < n \leqslant 3$
**g** $-4 < n < -1$
**h** $3 < n \leqslant 4$
**i** $-1 \leqslant n \leqslant 5$
**j** $-4 \leqslant n \leqslant -2$

**2** Simplify the inequalities and show the solutions as a range on a number line.

**a** $-3 < x + 1 < 4$
**b** $-2 < x - 2 \leqslant 0$
**c** $2 \leqslant 2x < 8$
**d** $-5 \leqslant 2x + 1 < 7$
**e** $0 < 2x - 4 \leqslant 3$
**f** $-11 \leqslant 3x - 5 < 4$
**g** $-1 < 3 + 2x < 5$
**h** $9 < 4x - 1 \leqslant 21$

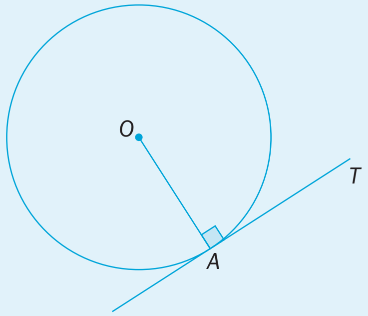
**3** For each pair of inequalities, shade the required region on a graph.
You may need to simplify the inequalities first.

**a** $x > 1, y > 2$
**b** $x \leqslant 2, y > 0$
**c** $x > -1, y \leqslant 3$
**d** $x \geqslant 0, y \geqslant 2$
**e** $x + 2 > 0, y - 1 > 0$
**f** $2x \leqslant 5, 3y > 6$
**g** $2x + 1 < 0, 3y - 1 \geqslant 0$
**h** $4x - 3 > 9, 5 + 2y \leqslant 3$
**i** $3 + 2x \leqslant 1, 3 + 2y \leqslant -2$
**j** $2 + 3x < x, 5 + 2y > y$
**k** $5x - 4 > 3x, 2(y - 1) < 3$
**l** $2\,(x + 3) < 10, 3(y - 1) > 2y$

# 0.8 Circle theorems

**Theorem:** *The angle between a tangent to a point on a circle and a radius at the same point is a right angle.*

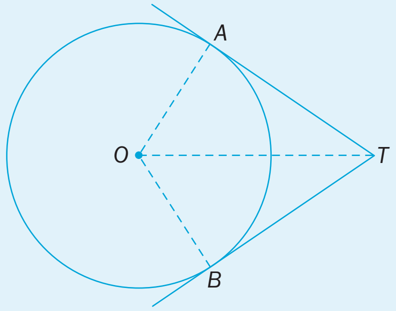
If the centre of the circle is at the point $O$ and $A$ is a point on the circumference of the circle, the tangent drawn to the circle at point $A$ makes an angle of 90° with the radius, $OA$.



Angle $OAT = 90^\circ$

**Theorem:** *Tangents drawn from two points on the circumference of a circle intersect at a point and are of the same length.*

Let $T$ be the point of intersection of the tangents drawn from the points $A$ and $B$ on the circumference of a circle.



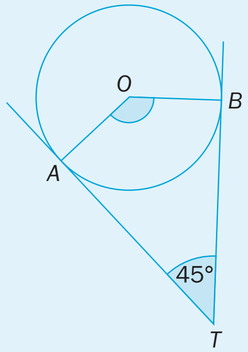
Angle $OAT$ = angle $OBT = 90^\circ$
Length $OA = OB$ (radii)
Length $OT$ is common to both triangles.

Triangles $OAT$ and $OBT$ are congruent.
$\therefore\ AT = BT$

EXAMPLE 1

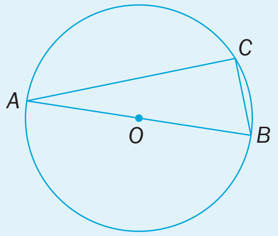
$AT$ and $BT$ are tangents to the circle, centre $O$.
Angle $ATB = 45^\circ$. Find the obtuse angle $AOB$.

Angle $OAT = OBT$ $(= 90^\circ)$
(radius and tangent)
Angle sum of a quadrilateral is 360°.
Obtuse angle $AOB = 360^\circ - (90^\circ + 90^\circ + 45^\circ)$
$= 135^\circ$



**Theorem:** *The angle in a semicircle is a right-*

The line $AOB$ is a diameter of the circle where $A$ and $B$ are points on the circumference.
$C$ is another point on the circumference.
Angle $ACB = 90^\circ$



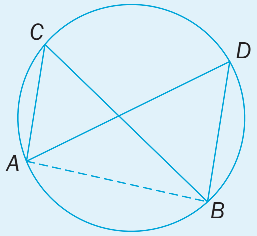
Angle $ACB$ is called 'the angle in a semicircle'.

**Theorem:** *Angles drawn on the same chord in the same segment of a circle are equal.*

The ends of the chord $AB$ lie on the circumference of a circle. The chord divides the circle into a major segment (the larger one) and a minor segment.

The two angles $ACB$ and $ADB$ are in the same segment.

Angle $ACB$ = Angle $ADB$

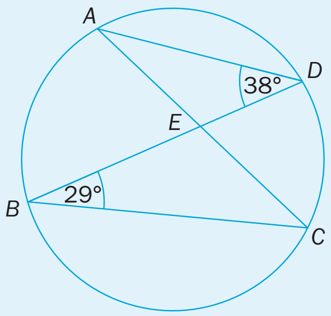


EXAMPLE 2

The points $A$, $B$ , $C$ and $D$ lie on the circumference of a circle. The lines $AC$ and $BD$ intersect at $E$. Angle $ADB = 38^\circ$ and $CBD = 29^\circ$. Find angles $CAD$ and $AED$.

Draw the chord from $C$ to $D$ making two segments:
Angle $CAD$ = angle $CBD$ (angles on the same chord in the same segment)
$\therefore$ angle $CAD = 29^\circ$

In triangle $AED$
Angle $AED = 180^\circ - (29^\circ + 38^\circ)$ (angle sum of triangle)
Angle $AED = 113^\circ$

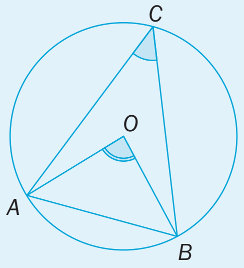


**Theorem:** *The angle subtended by a chord at the centre of a circle is twice the angle subtended by the same chord at the circumference.*

Let the chord of the circle, with centre $O$, be $AB$.

The angle at the centre is $AOB$ and the angle at the circumference is $ACB$; both angles are subtended by the chord $AB$.

Angle $AOB = 2 \times$ angle $ACB$



**Theorem:** *The angle between a chord and a tangent is equal to the angle subtended by the chord in the opposite segment.*

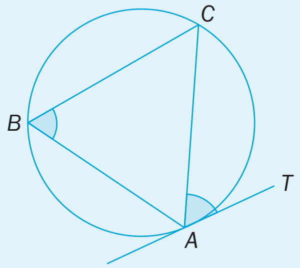
This theorem is called 'the alternate segment theorem'.

Let the tangent $AT$ be drawn to the circle at the point $A$.

The angle between the chord $AC$ and the tangent $AT$ is the angle $CAT$.

The angle in the alternate segment is the angle $ABC$.

Angle $CAT =$ angle $ABC$



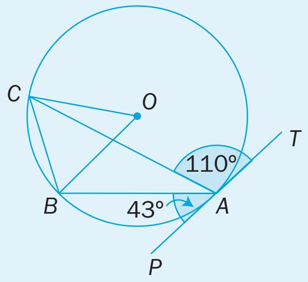
EXAMPLE 3

In the diagram, $PAT$ is a tangent to the circle, centre $O$. $B$ and $C$ are points on the circumference. Angle $CAT = 110°$ and angle $BAP = 43°$. Find angles $COB$ and $ABC$.

Angle $BAC = 180° - (110° + 43°)$ (angle on a straight line)
$= 27°$

Angle $COB = 2 \times$ angle $BAC$ (angle at centre twice angle at circumference)
$= 54°$

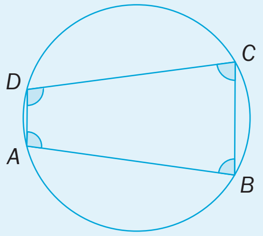
Angle $ABC =$ angle $CAT$ (alternate segment theorem)
so angle $ABC = 110°$



**Theorem:** *Opposite angles of a cyclic quadrilateral sum to*

In a **cyclic quadrilateral** the four vertices lie on the circumference of a circle.

Angles $A + C = 180°$
Angles $B + D = 180°$



EXAMPLE 4

Prove that the opposite angles of a cyclic quadrilateral are supplementary.

Supplementary angles add up to 180°.

Let the angle at $B$ be $p$ and the angle at $D$ be $q$.

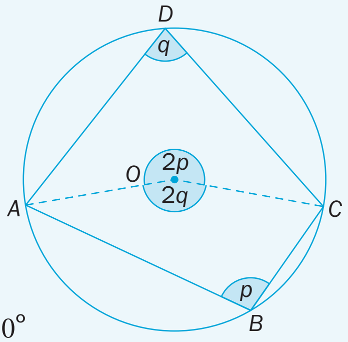
Using the theorem that the angle at the centre is twice the angle at the circumference,

obtuse angle $AOC = 2q$
reflex angle $AOC = 2p$

The sum of these two angles is 360° (angles at a point).

Hence $2p + 2q = 360°$
$\therefore\ p + q = 180°$

So opposite angles of a cyclic quadrilateral are supplementary.



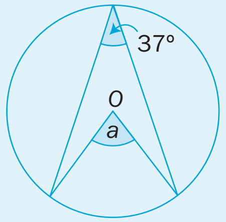
Using chord DB you can prove that angles $A + C = 180°$

## Exercise 0.8

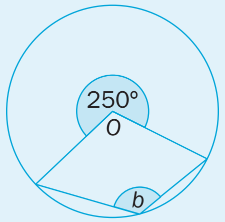
The diagrams in this Exercise are not drawn to scale.

**1** In these diagrams, $O$ is the centre of the circle. Find the missing angles.

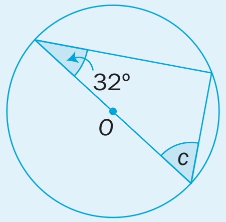
**a**



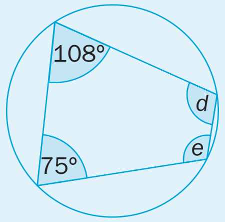
**b**



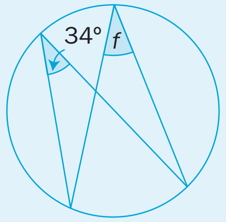
**c**



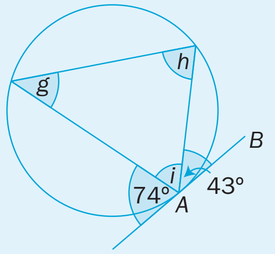
**d**



**e**

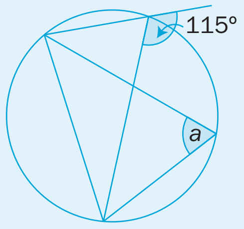


**f** *AB* is a tangent.

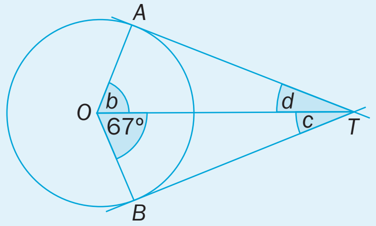


**2** In these diagrams, *O* is the centre of the circle.
Find the missing angles.

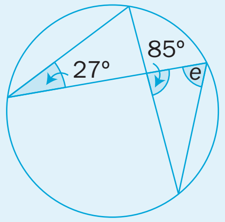
**a**



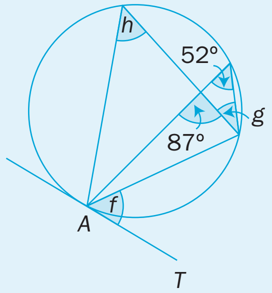
**b** *AT* and *BT* are tangents.



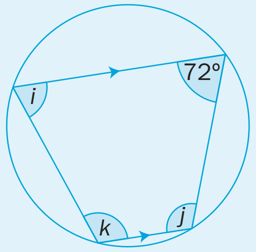
**c**



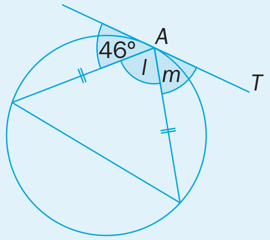
**d** *AT* is a tangent.



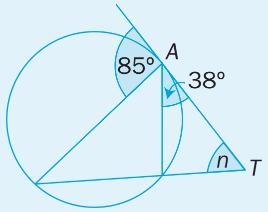
**e**



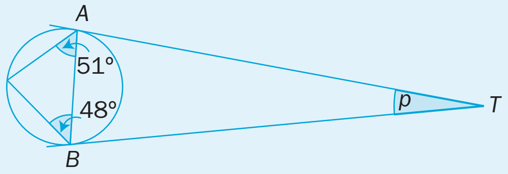
**f** *AT* is a tangent.



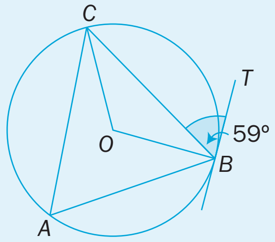
**g** *AT* is a tangent.



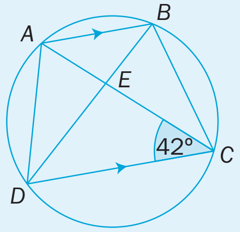
**h** *AT* and *BT* are tangents.



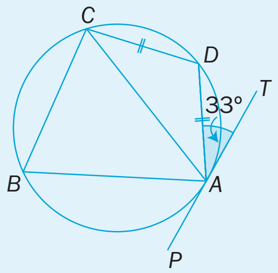
**3** **a** Points *A*, *B* and *C* are on the circumference of the circle, centre *O*.
*BT* is a tangent to the circle at point *B*.
If the angle *CBT* = 59°, find the size of the angle *COB*.



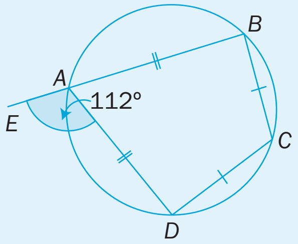
**b** The points *A*, *B*, *C* and *D* are on the circumference of a circle.
Chord *AB* is parallel to chord *CD*.
Angle *ACD* = 42°.
*AC* and *BD* intersect at *E*.
Find angles **i** *DBA*
**ii** *BDC*
**iii** *BEC*.



**c** The points *A*, *B*, *C* and *D* are on the circumference of a circle.
*PAT* is a tangent to the circle at point *A*.
*AD* = *CD*.
Angle *DAT* = 33°.
Find the angles **i** *ACD* **ii** *CDA*
**iii** *CBA* **iv** *CAP*.



**d** The points *A*, *B*, *C* and *D* lie on the circumference of a circle.
*AB* = *AD* and *BC* = *CD*.
*BA* is produced to point *E* and angle *EAD* = 112°.
Find the sizes of the angles
**i** *BAD* **ii** *ABC*
**iii** *BCD* **iv** *CDA*.



**4** **a** Prove that angles drawn on the same chord in the same segment of a circle are equal.

**b** Prove that the angle at the centre of a circle is twice the angle at the circumference when the angles are subtended by the same chord.

**c** Prove the alternate segment theorem.

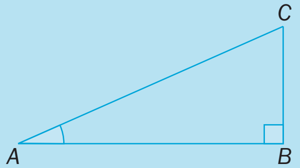
# 0.9 Trigonometry

For the right-angled triangle $ABC$ the three trigonometric ratios are

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{AC}$$

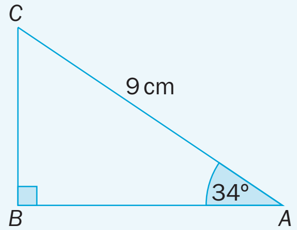
$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{BC}{AB}$$



Pythagoras' theorem states that
$AB^2 + BC^2 = AC^2$

**EXAMPLE 1**

In the right-angled triangle $ABC$,
side $AC = 9$ cm and angle $A = 34°$.
Find the lengths of sides $AB$ and $BC$.



Use the cosine ratio to find $AB$:

$$\cos A = \frac{adj}{hyp} = \frac{AB}{AC}$$

$$\cos 34° = \frac{AB}{9}$$

$$9 \times 0.8290 = AB$$
$$AB = 7.46 \text{ cm}$$

Use the sin ratio to find $BC$:

$$\sin A = \frac{opp}{hyp} = \frac{BC}{AC}$$

$$\sin 34° = \frac{BC}{9}$$

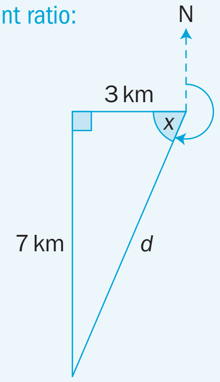
$$9 \times 0.5592 = BC$$
$$BC = 5.03 \text{ cm}$$

The trigonometric ratios may also be applied to more practical questions. A diagram will help you to visualise the problem.

**EXAMPLE 2**

Tony walks 3 km in a westerly direction and then turns south and walks for a further 7 km. What is his bearing and how far is he from his starting point?

Work out the angle in the triangle first, using the tangent ratio:



$$\tan x = \frac{\text{opposite}}{\text{adjacent}} = \frac{7}{3}$$

$$\tan x = 2.333\ldots$$
$$x = 66.8°$$

Subtract 66.8° from 270°:

His bearing is 203.2°.

The three-figure bearing is taken from the northerly direction.

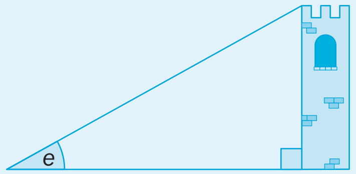
Calculate the distance using Pythagoras' theorem:

$$d^2 = 3^2 + 7^2 = 9 + 49 = 58$$
$$d = 7.62$$

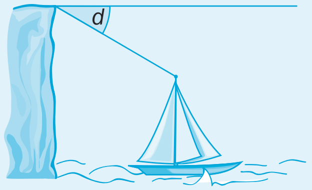
His distance from the starting point is 7.62 km.

### Angles of elevation and depression

Angles of elevation and depression are measured from the horizontal.



The angle of **elevation** of the top of the tower is $e$.

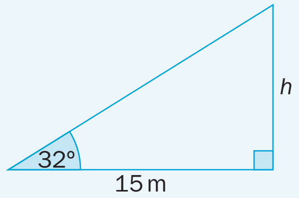


The angle of **depression** of the top of the mast is $d$.

**EXAMPLE 3**

From an observation point 15 m from the base of the tower the angle of elevation of the top of tower is 32°.
Find the height of the tower.

Draw a diagram:



Using the tangent ratio:

$$\tan 32° = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{15}$$

where $h$ is the height of the tower

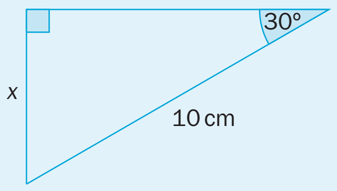
$$15 \times \tan 32° = h$$
$$15 \times 0.625 = h$$
$$h = 9.373$$

The height of the tower is 9.37 m.

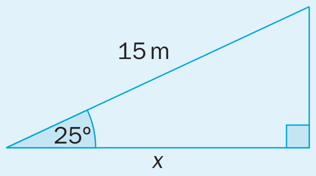
## Exercise 0.9

**1** Work out the value of $x$ in each of these triangles.
Where appropriate give your answers correct to 3 s.f.
Diagrams are not drawn to scale.

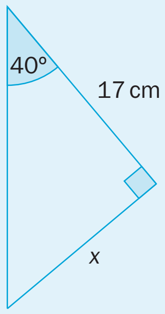
**a**



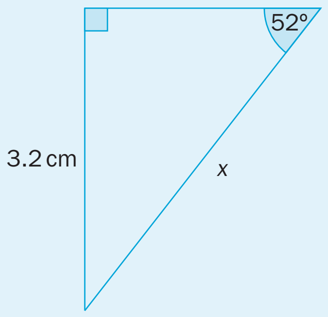
**b**



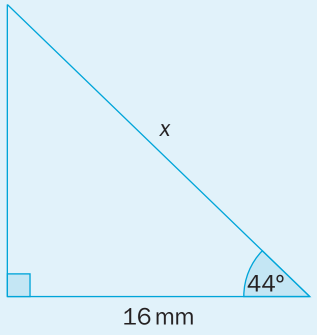
**c**



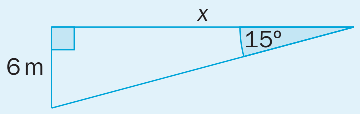
**d**



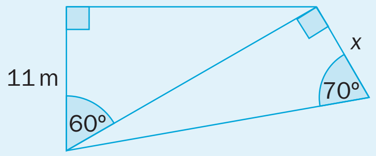
**e**



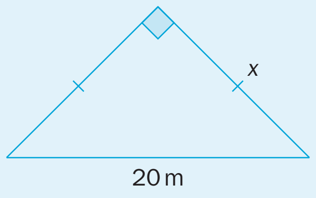
**f**



**g**

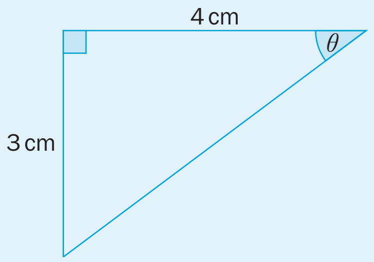


**h**

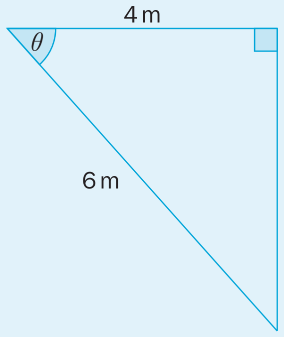


**2** Find the value of $\theta$ in each part.
Where appropriate give your answers correct to 3 s.f.
Diagrams are not drawn to scale.

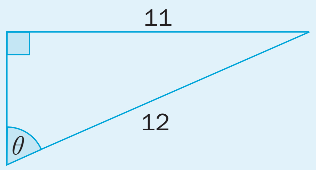
**a**



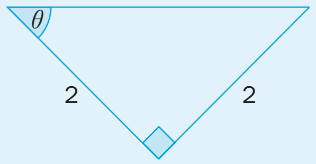
**b**



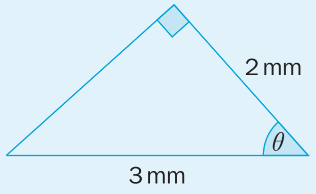
**c**



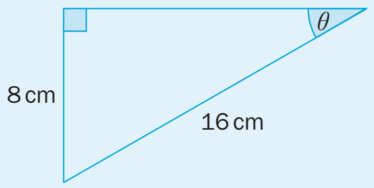
**d**



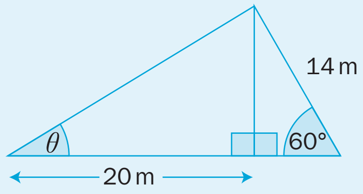
**e**



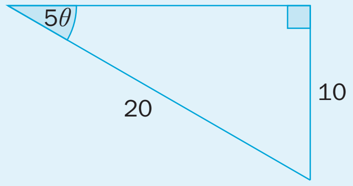
**f**



**g**



**h**



**3** Draw a diagram for each part.
Give your answers correct to three significant figures or two decimal places.

**a** Sonia is measuring the angle of elevation of the top of a flagpole, 9 m high.
Her eyes are 1.3 m above ground level and she is at a distance of 5 m from the flagpole.
What is the angle of elevation of the top of the flagpole from Sonia's eye level?

**b** From a starting point, *A*, Graham cycles 3.2 km east and then 1.2 km north to reach the base of a tower.
The angle of elevation of the top of the tower from his starting point is 1.85°.
Find the height of the tower in metres.

**c** The vertical angle of a right circular cone is 37° and the vertical height of the cone is 9.8 cm.
What is the base radius of the cone?

**d** The longest diagonal in a cuboid measures 17.8 cm and makes an angle of 21° with the bottom face of the cuboid.
Find the height of the cuboid.
If one of the base edges of the cuboid measures 3.45 cm find the length of the other base edge.

**e** The radius of a circle is 23 cm and an arc of the circle subtends an angle of 73° at the centre.
Find the length of the chord joining the ends of the radii.

**f** A simple pendulum swings between two points 35 cm apart on the same horizontal level.
The angle at the centre of oscillation is 35°.
Find the length of the string supporting the pendulum.

**g** Alex walks 4 km in a north-east direction to a point *A*.
From the same starting point as Alex, Bryony walks 7 km in a south-east direction to a point *B*.
Find the bearing of point *A* from point *B*.

**4** Draw a diagram for each part.

**a** The points *A* (1,3) and *B* (3,8) are joined by a straight line.
Find the angle the extended line *AB* makes with the *x*-axis.

**b** A pyramid has a square base of side 9.5 cm and a vertical height of 8 cm.
Find the angle a face of the pyramid makes with the base.

**c** From a point *A* the angle of elevation of the top of a mast is 21° and from a point *B* the angle is 16°.
The base of the mast and the points *A* and *B* are in a straight line on horizontal ground with *A* and *B* on opposite sides of the mast.
If the height of the mast is *h* metres find the distance *AB* in terms of *h*.

**d** A cylindrical tube of base circumference 27 cm is of height 12 cm. A stick of negligible thickness just fits inside the tube at an angle.
Find the length of the stick and the angle it makes with the horizontal.

**e** A point *A* is on a bearing of 063° from a fixed point, *P*, and a point *B* is on a bearing of 133° from *P*.
In this position *A* is due north of *B*.
The line joining *A* to *B* is 5.35 km from *P* at the nearest point.
Find the distance *AB*.

## Review 0

**1** Simplify these expressions expanding first where necessary.

**a** $3b - 2c + 5c - 4b$
**b** $2(a - 2b) + 3(b - 2a)$
**c** $p(q - 1) + q(p - 1)$
**d** $a + b - 2(a - b)$
**e** $(x + 2)(x - 5)$
**f** $(2x - 3)(2x + 3)$
**g** $(y - 1)(3y - 4)$
**h** $2(3y - 1)(y + 2)$
**i** $x^2(x - 1)(x + 3)$
**j** $(x^2 - 1)(x + 3)$
**k** $(x - 1)(x + 1)(x + 2)$
**l** $(x + 2)(x + 3)(x + 4)$

**2** Given $a = 4$, $b = -5$ and $c = 2$ find the values of these expressions.

**a** $2b + 3a$
**b** $(2b)^2$
**c** $\frac{c}{a} - \frac{a}{b}$
**d** $\sqrt{(b^2 - a^2)}$

**3** Evaluate these formulae.

**a** $L = \frac{T^2 g}{4\pi^2}$ find $L$ when $T = 100$, $g = 9.8$ and $\pi^2 = 10$

**b** $V = \frac{4}{3}\pi r^3$ find $r$ when $\pi = 3$ and $V = 108$

**c** $I = \frac{PTR}{100}$ find $R$ when $I = 50$, $P = 1000$ and $T = 2.5$

**4** Solve these equations.

**a** $4(2t - 1) + 3(1 - 2t) = 0$
**b** $0.7(4p - 3) = 1.4$
**c** $\frac{3r}{2} + \frac{5r}{6} = 7$
**d** $\frac{2}{5y + 1} = 3$

**5** Draw the graphs of these equations for $-2 \leqslant x \leqslant 4$.

**a** $2y = -x + 4$
**b** $y = x^2 - 5x + 3$
**c** $y = -\mathrm{f}(x)$ when $\mathrm{f}(x) = -2x + 1$
**d** $y = 2\mathrm{f}(x)$ when $\mathrm{f}(x) = x - 2$

**6** Solve these simultaneous equations using an algebraic method.

**a** $2x + 3y = 7$
$5x - 3y = -14$

**b** $5y + 4x = 2$
$9x + 5y = -8$

**c** $3y - 4x = -8$
$y - 2x = -3$

**d** $5x - 2y = 15$
$4y - x = 6$

**e** $2x - 3y = 5$
$3x - 4y = -1$

**f** $5x + 4y = 0$
$3x - 2y = 11$

**g** $6 - 7x = 4y$
$5y = 2x + 29$

**h** $10x = 3y - 2$
$7x = 2y - 1$

**7** Use a graphical method to solve these simultaneous equations.

**a** $y = x - 2$, $2y - x = 0$
**b** $\frac{x}{2} + \frac{y}{4} = 1$, $y = 2x - 1$
**c** $2y + 3x = 1$, $2y - 3x = -1$
**d** $y + 4 = 5x$, $x + y = 2$

**8 a** Show the solutions of these inequalities on a number line.

**i** $-1 < x + 1 \leqslant 4$
**ii** $2x < 5$, $3x - 1 \geqslant -7$

**b** Show the solutions of these inequalities as a region on a graph. Shade the required regions.

**i** $x + 3 > 4$ and $2y - 3 \leqslant 5$
**ii** $3y + 4 \geqslant 2y + 3$ and $-1 < x < 3$

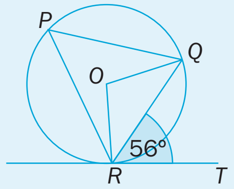
**9 a** The point $(x, y)$ has integer coordinates and lies in the region defined by $1 < x < 3$ and $2 < y < 4$.
State the values of $x$ and $y$.

**b** Find the largest integer $x$-value and $y$-value which lies in the region defined by $x < 2$ and $y < 4$.

**c** The sum of two positive integers, $a$ and $b$, is less than 13, and the difference between the two numbers is less than 3.
Investigate possible values of $a$ and $b$ if $a > b$.

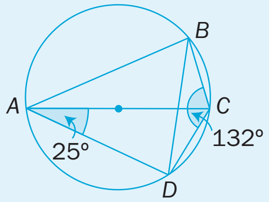
**d** Given that $x < 2y$ shade the region on a graph to indicate where the inequality is located.

**10** $P$, $Q$ and $R$ are points on a circle.
$O$ is the centre of the circle.
$RT$ is the tangent to the circle at $R$.
Angle $QRT = 56°$.



**a** Find
**i** the size of angle $RPQ$
**ii** the size of angle $ROQ$.

$A$, $B$, $C$ and $D$ are points on a circle.
$AC$ is a diameter of the circle.
Angle $CAD = 25°$ and angle $BCD = 132°$.



**b** Calculate
**i** the size of angle $BAC$
**ii** the size of angle $ABD$.

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**11** Draw diagrams and use them to solve the following problems.

**a** Find the vertical height of an equilateral triangle whose sides are of length 7.5 cm.

**b** The diagonals of a rhombus are 12 cm and 10 cm.
Find the size of the angles at the vertices of the rhombus.

# 0

# Exit

## Summary

| | Refer to |
|---|---|
| Algebraic expressions can be simplified when there are like terms present. | 0.1 |
| You factorise a quadratic expression by writing it as a product of its factors. | 0.1 |
| A formula can be evaluated by substituting the given values into it and working out the result. | 0.2 |
| To change the subject of a formula you rewrite the equation to express the named letter in terms of the other variables. | 0.2 |
| You can solve a linear equation by rearranging its terms. Take care in dealing with signs. | 0.3 |
| The graph of a linear equation is a straight line. | 0.4 |
| You can use Pythagoras' theorem to find the midpoint and length of a line segment. | 0.5 |
| Simultaneous equations can be solved by elimination, substitution, or by drawing a graph. | 0.6 |
| Inequalities are used to define a range or a region on a graph. | 0.7 |
| Circle theorems can be applied to find angles within circles. | 0.8 |
| Pythagoras' theorem can be used in a right-angled triangle if two sides are known. | 0.9 |
| The trigonometric ratios can be used in a right-angled triangle if one side and an angle or two sides are known. | 0.9 |